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Valve effect of inhomogeneities on anisotropic wave propagation

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A recent investigation of hydromagnetic waves in a rotating fluid has revealed certain 'valve'-like critical levels associated with each wave which can be effectively penetrated *from one side only*. This effect is illustrated in the present paper by means of two further examples, namely (*a*) the propagation of hydromagnetic–gravity waves in a non-uniform magnetic field, and (*b*) the propagation of internal gravity waves in a wind which, though unidirectional, is both horizontally and vertically sheared.

1. Introduction

The propagation of plane hydromagnetic waves in a uniformly rotating fluid permeated by a co-rotating but non-uniform magnetic field has been considered in a recent paper (Acheson 1972*a*, hereafter referred to as A). The spatial variations of the magnetic field were assumed to be such that by choosing a suitably oriented rectangular Cartesian co-ordinate system (x, y, z) rotating with both fluid and field the latter could be expressed in the form $\mathbf{B}_0 = \{B_x(z), B_y(z), 0\}$, the z axis not necessarily coinciding with the angular velocity vector

$$\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z).$$

In addition, the fluid was assumed non-dissipative, incompressible and of uniform density. When $\mathbf{\Omega} = 0$ hydromagnetic waves propagate along the lines of force and there is in consequence no energy flux in the z direction. In a rotating system, however, the action of the Coriolis force permits hydromagnetic wave propagation *across* field lines (Lehnert 1954). In A it was demonstrated that associated with each wave (characterized by a frequency ω and wavenumber components k and l in the x and y directions respectively) there are 'critical levels' $z = z_c$ (at which the quantity $|B_x k + B_y l|$ assumes one of two special values depending *inter alia* on the wave in question) which the wave may effectively penetrate *one way only* (i.e. either from the side $z < z_c$ or from the side $z > z_c$). If the wave approaches one of its critical levels from the 'wrong' side very little of its energy is transmitted or reflected, most of it instead being trapped in the immediate vicinity of the critical level.

This 'valve' effect may be described particularly simply for a class of 'slow' hydromagnetic waves of interest in connexion with the dynamics of the earth's liquid core (for references see Acheson & Hide 1973; Hide & Stewartson 1972; Roberts & Soward 1972). Associated with each 'slow' wave there is then (as in the example in §2; see equation (2.3)) only one critical value of the quantity $|B_x k + B_y l|$, and the wave may penetrate its critical level only if its local speed of propagation W in the z direction is such that

$$W\Omega_z(\Omega_x k + \Omega_y l)\,\omega < 0 \tag{1.1}$$

(Acheson 1972*a*).

The purpose of the present paper is to point out that this 'valve' effect may, in principle at least (cf. §4), be a common feature of inhomogeneous systems supporting anisotropic wave propagation, *provided only that a certain inherent asymmetry is present.* The required asymmetry in the above system, for example, is present if the rotation vector $\mathbf{\Omega}$ is neither parallel to nor perpendicular to the direction z of non-uniformity, as evinced by equation (1.1). Now suppose instead that both the fluid and the field $\mathbf{B}_0 = \{B_x(z), B_y(z), 0\}$ are *stationary* but that the fluid is stably *stratified* in the direction of gravity $\mathbf{g} = (g_x, g_y, g_z)$. Similar effects then arise, and these are discussed in §2. A further example of valve-like critical-level behaviour is provided in §3, where we consider disturbances to the unidirectional shear flow $\mathbf{U} = \{U_x(z), 0, 0\}$ of a stratified fluid under gravity $\mathbf{g} = (0, g_y, g_z)$. With $g_y \neq 0$ this constitutes a simple, if slightly artificial, variant of the problem initially studied by Bretherton (1966) and Booker & Bretherton (1967).

When the typical wavelengths λ involved are very short compared with the length scale L characterizing the inhomogeneities of the system the propagation of localized disturbances through the fluid is conveniently viewed in terms of the motion of wave 'packets', each of which possesses a reasonably well-defined frequency ω and wavenumber vector $\boldsymbol{\kappa} = (k, l, m)$. When, by hypothesis, the local properties of the system depend only on z (say), the set of quantities ω, k and l acts as a 'label' for the packet, while the z component m of the wavenumber changes as the packet moves, its value at any level being given in terms of the local properties of the medium by the dispersion relation

$$m = m(\omega, k, l, z), \tag{1.2}$$

which may alternatively be written as

$$\omega = \omega(\boldsymbol{\kappa}, z) \tag{1.3}$$

(see equation (1.5)). Further, the velocity with which any particular packet moves through the fluid is given in terms of the local value of m and z by the familiar expression

$$\mathbf{u}_g = (u_g, v_g, w_g) \equiv \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) \tag{1.4}$$

for the group velocity, obtained by differentiating (1.3). On combining (1.2) and (1.4) the propagation velocity $\mathbf{u}_g$ of any packet may thus be expressed in terms of z, provided only that the local dispersion relation (1.3) has been established.

For hydromagnetic waves in a non-dissipative incompressible fluid whose undisturbed state is one of rest the relation (1.3) takes the form

$$\omega^2 = (\mathbf{V}\,.\,\boldsymbol{\kappa})^2 + (\mathbf{N}\times\boldsymbol{\kappa})^2/\boldsymbol{\kappa}^2 \tag{1.5}$$

(see, for example, Chandrasekhar 1961; Hide 1969). Here the Brunt–Väisälä frequency

$$\mathbf{N} \equiv \frac{\mathbf{g}}{|\mathbf{g}|}\left(\frac{\mathbf{g}\,.\,\nabla\rho_0}{\rho_0}\right)^{\frac{1}{2}} = \left\{\left(\frac{g_x}{\rho_0}\frac{\partial\rho_0}{\partial x}\right)^{\frac{1}{2}}, \left(\frac{g_y}{\rho_0}\frac{\partial\rho_0}{\partial y}\right)^{\frac{1}{2}}, \left(\frac{g_z}{\rho_0}\frac{\partial\rho_0}{\partial z}\right)^{\frac{1}{2}}\right\} \tag{1.6}$$

(which is, in this paper, constant by hypothesis) acts as a convenient measure of the buoyancy of the fluid, and the Alfvén velocity

$$\mathbf{V} \equiv \{B_x(z), B_y(z), 0\}/(\mu\bar{\rho}_0)^{\frac{1}{2}} \tag{1.7}$$

acts as a convenient measure of the local magnetic field (μ, ρ_0 and $\bar{\rho}_0$ being the magnetic permeability, density and mean density of the fluid respectively).

When $\mathbf{N} = 0$ equation (1.5) reduces to the dispersion relationship for plane Alfvén waves (see, for example, Alfvén & Fälthammar 1963, p. 78), for which Lorentz forces alone provide the restoring torques on individual fluid elements. When $\mathbf{V} = 0$ equation (1.5) reduces to the dispersion relationship for internal gravity waves (see, for example, Yih 1969), for which buoyancy forces alone provide the restoring torques.

The requirements for the validity of (*a*) the neglect, as implied by the use of the *mean* density in (1.7), of ray path distortion due to density variations (which are in the direction of gravity) as compared with that due to magnetic field variations (which, by definition, take place in the z direction) and (*b*) the 'Boussinesq approximation' used in the derivation of (1.5) are both met if, in addition to the 'short-wavelength' restriction $\lambda \ll B/|\nabla B|$ implicit in the wave-packet approach,

$$|\nabla\rho_0|/\rho_0 \ll |\nabla B|/B. \tag{1.8}$$

Similar remarks pertain to the non-hydromagnetic example in §3, with the magnetic field in (1.8) being then replaced by the wind U.

While the wave-packet approach provides an attractively simple picture of the various propagation properties it is an unfortunate fact that it predicts, in both the systems investigated in this paper (as in those studied by Bretherton (1966) and Acheson (1972*a*)), an indefinite increase in the wave energy density of a packet which is in the process of being 'captured' at its critical level, so that the assumption of small wave amplitude on which the whole analysis is based ultimately breaks down. Results obtained by this formalism must therefore be regarded only as *suggestive* of what to look for in more careful studies of the dynamics in the immediate neighbourhood of the critical levels (such as those of Booker & Bretherton (1967) and §3 of A). Accordingly in both §§2 and 3 the wave-packet approach is followed by a brief presentation of the results obtained from such analyses, which do not involve the 'short-wavelength' approximation.

2. Hydromagnetic–gravity waves in a sheared magnetic field

Consider now the propagation of a wave packet under the influence of both Lorentz and buoyancy forces in the system outlined above. Any wave packet (identified by particular values of ω, k and l) then has a z component of propagation velocity

$$w_g = \frac{m\{(V_x k + V_y l)^2 + N_x^2 + N_y^2 - \omega^2\} - N_z(N_x k + N_y l)}{\omega\,(k^2 + l^2 + m^2)} \tag{2.1}$$

and the local value of m is related to that of $\mathbf{V}$ by

$$\{(V_x k + V_y l)^2 + N_x^2 + N_y^2 - \omega^2\} m^2 - 2N_z(N_x k + N_y l)\, m + (k^2 + l^2)\{(V_x k + V_y l)^2 + N_z^2 - \omega^2\} + (N_x l - N_y k)^2 = 0 \tag{2.2}$$

(see (1.2)–(1.5)), an equation quadratic in m. Thus, as the packet approaches a level $z = z_c$ where the quantity $(V_x k + V_y l)^2$ takes the special value

$$\{V_x(z_c)\, k + V_y(z_c)\, l\}^2 = \omega^2 - N_x^2 - N_y^2, \tag{2.3}$$

one value of m given by (2.2) increases in magnitude indefinitely as $|z-z_c|^{-1}$ (provided $[d\{V_x k + V_y l\}^2/dz]_{z=z_c} \neq 0$), being given asymptotically by

$$m_1\{(V_x k + V_y l)^2 + N_x^2 + N_y^2 - \omega^2\} = 2N_z(N_x k + N_y l). \tag{2.4}$$

The z velocity component of the packet evidently takes the asymptotic form

$$w_g = N_z(N_x k + N_y l)/\omega m_1^2 \tag{2.5}$$

(see equation (2.1)), so that when $|z-z_c|$ is small the level of the packet changes according to the equation

$$dz/dt = a(z-z_c)^2, \tag{2.6}$$

where a is a constant. This may be integrated to give

$$z - z_c = -1/(at+b), \tag{2.7}$$

where b is another constant, and the packet thus slows down in such a way that *it never reaches its critical level in a finite time.* It is therefore neither transmitted nor reflected, but instead effectively 'captured' and constrained to propagate almost along the lines of force there, as evinced by the asymptotic relations for its other propagation components:

$$\omega(u_g, v_g) = (V_x k + V_y l)\,(V_x, V_y). \tag{2.8}$$

The other value of m tends to a finite value m_2 (obtained by neglecting the quadratic term in (2.2)) as the critical level is approached, and the corresponding z component of the group velocity tends to a non-zero value:

$$w_g = -N_z\,(N_x k + N_y l)/\omega(k^2 + l^2 + m_2^2). \tag{2.9}$$

A wave packet approaching its critical level is therefore either transmitted or captured there according to the criterion

$$w_g N_z(N_x k + N_y l)\,\omega \begin{cases} < 0 & \text{for transmission,} \\ > 0 & \text{for capture.} \end{cases} \tag{2.10}$$

Note the striking similarity in form between the criterion (2.10) and equation (1.1), which gives the transmission criterion for a 'slow' hydromagnetic wave packet in a rapidly *rotating* fluid of *uniform density*. The criterion (1.1) is obtained by formally replacing $\mathbf{N}$ in (2.10) by $\mathbf{\Omega}$.

While the valve-like critical levels above are the only ones strictly germane to the subject of this paper there are other levels at which waves cease to propagate across field lines. The roots of (2.2) are real only if

$$N_x^2 + N_y^2 + N_z^2 \geqslant \omega^2 - (V_x k + V_y l)^2 \geqslant (N_x l - N_y k)^2/(k^2 + l^2). \tag{2.11}$$

If a particular packet propagates towards a level $z = z_r$ where the magnetic field takes either of the above limiting values its z component of propagation speed decreases as $|z-z_r|^{\frac{1}{2}}$. It therefore reaches such a level in a *finite* time (cf. (2.6) and (2.7)) and is reflected.

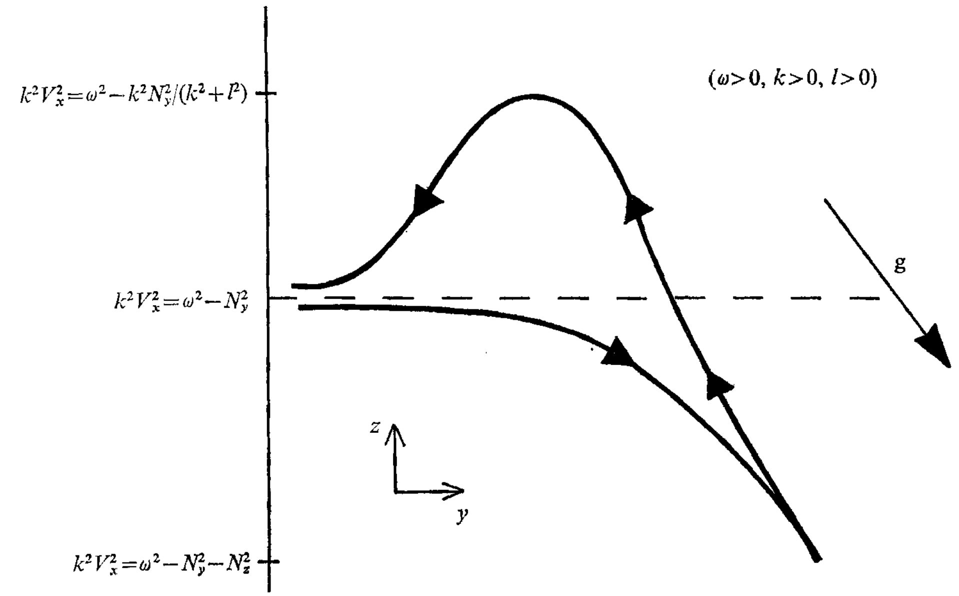

FIGURE 1. Schematic diagram illustrating the 'valve' effect for a hydromagnetic wave packet propagating in a stratified fluid.

Figure 1 provides a sketch (based on careful investigation of the ray path slope $dz/dy = w_g/v_g$ at various key points, e.g. $z = z_c$ and $z = z_r$) of the *projection in the y, z plane* of a complete ray path for the case $N_x = 0$ (cf. McKenzie 1973). The magnetic field, measured by V_x, is directed out of the paper and increases with z. Gravity $\mathbf{g} = (0, g_y, g_z)$ acts in the y, z plane, and to view the system from a more conventional standpoint one may turn the sketch clockwise through about 45°. The path drawn is that appropriate to a wave with ω, k and l all positive. If a localized disturbance with these properties is generated at any level z and its initial direction of propagation along the z axis is specified then its subsequent motion can be traced out by following the arrows in the sketch. Thus, such a wave generated somewhere below its critical level and initially moving downwards is bent toward the vertical and undergoes a reflexion. While both horizontal and vertical propagation speeds approach zero (the former as $|z - z_r|$ and the latter as $|z - z_r|^{\frac{1}{2}}$) the packet's propagation speed u_g *out of the paper* approaches

$$(\omega^2 - N_y^2 - N_z^2)/\omega k;$$

thus what appears in projection to be cusp-like reflexion would not appear so in three dimensions. The packet is then bent away from the vertical and penetrates its critical level from below (see (2.10), setting $N_x = 0$). After undergoing another reflexion, being unable to penetrate its critical level from above, it is finally captured there.

We note that a wave packet with ω and k positive but l *negative* follows a similar path to that illustrated in figure 1 but proceeds in a direction *opposite* to that shown by the arrows (as one may check by noting from (2.10) that when $N_x = 0$ a change of sign of l changes the sense of the valve effect).

While the elementary and essentially *kinematic* considerations above (cf. Lighthill 1965), on which figure 1 is based,† provide perhaps the simplest and most concise way of illustrating the main result of this paper it would be unwise to place too much confidence in the method's predictions of events at the levels $z = z_r$, particularly when $\lambda \ll L$ is not satisfied, until more detailed studies including considerations of *amplitude* changes as the packet propagates (cf. remarks at end of §1) have been carried out.

The essential features of the *valve* effect have been confirmed, however, *even when* λ/L *is only marginally less than unity*, by an analysis identical in form to that in §3 of A (which is hence omitted here, but see Acheson 1971). A wave approaching its critical level from a side which, on the wave-packet formalism, would result in capture (see equation (2.10)), is in fact partially transmitted, albeit with its associated wave energy flux in the z direction (which, as in A, is independent of z on either side of the critical level) cut in the process by a factor

$$\exp 4\pi \left| \frac{N_z(N_x k + N_y l)}{d(V_x k + V_y l)^2/dz} \right|_{z=z_c}. \tag{2.12}$$

Since $(V_x k + V_y l)^2 = \omega^2 - N_x^2 - N_y^2$ at this level, (2.12) typically represents a very severe attenuation provided only that $\lambda/L \lesssim 1$. On approaching from the other side, however, the same wave would be transmitted without attenuation.

3. Internal gravity waves in a shear flow

Suppose now that hydromagnetic effects are absent and that internal gravity waves propagate instead in a wind $\mathbf{U} = \{U_x(z), 0, 0\}$ sheared in some direction (z) other than the vertical. This is clearly compatible with the basic equation $\partial\rho/\partial t + \mathbf{u}.\nabla\rho = 0$ (expressing, in the absence of thermal conduction, conservation of density of individual fluid elements) only if the basic density gradient is perpendicular to the wind, i.e. $\nabla\rho_0 = (0, \partial\rho_0/\partial y, \partial\rho_0/\partial z)$. Since with such a wind profile the equilibrium is one of hydrostatic balance (i.e. $\nabla p_0 = \rho_0 \mathbf{g}$), the density gradient must be parallel to gravity and accordingly $\mathbf{g} = (0, g_y, g_z)$. Thus $N_x \neq 0$ is incompatible with the assumed (steady) wind profile, and the local dispersion relation simplifies to

$$(\omega - U_x k)^2 = \frac{(N_y m - N_z l)^2 + N_y^2 k^2 + N_z^2 k^2}{k^2 + l^2 + m^2} \tag{3.1}$$

(obtained by Doppler-shifting (1.5) and setting $\mathbf{V} = 0$). The z component of velocity of any packet is therefore

$$w_g = \frac{-m\{(\omega - U_x k)^2 - N_y^2\} - N_y N_z l}{(\omega - U_x k)(k^2 + l^2 + m^2)}. \tag{3.2}$$

Near a critical level where $(\omega - U_x k)^2 = N_y^2$ one root (m_1, say) of (3.1) increases as $|z - z_c|^{-1}$ and correspondingly

$$w_g \sim N_y N_z l/(\omega - U_x k) m_1^2, \tag{3.3}$$

† The following remarks pertain equally to the non-hydromagnetic example in §3 as illustrated in figure 2.

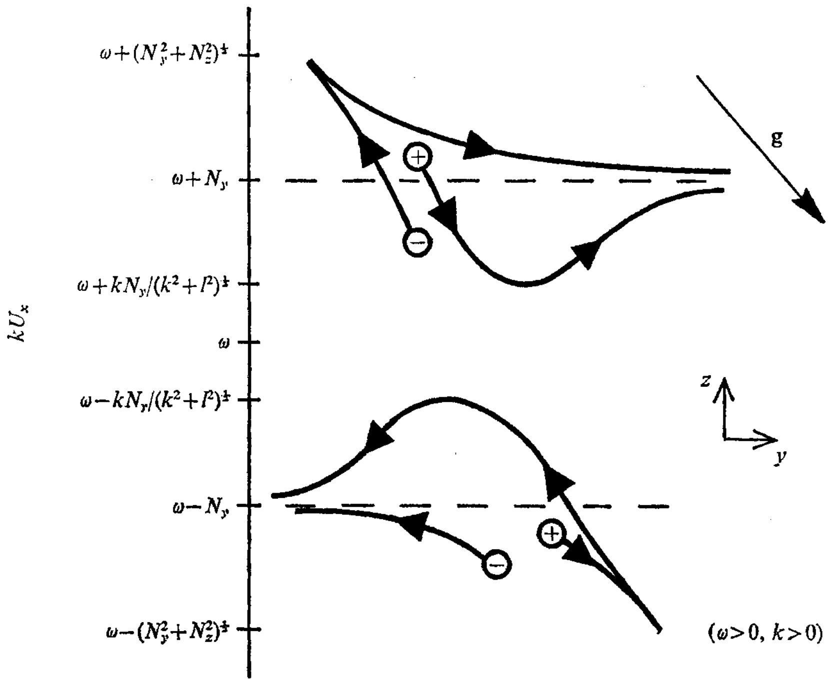

FIGURE 2. Schematic diagram illustrating the 'valve' effect for internal gravity wave packets propagating in a shear flow.

resulting in capture. In this case the packet in fact ceases to propagate relative to the flow at that level (cf. equation (2.8)). The other root tends to a finite value m_2 and so also does

$$w_g \sim -N_y N_z l/(\omega - U_x k)\,(k^2+l^2+m_2^2). \tag{3.4}$$

A wave packet approaching a critical level is therefore either transmitted or captured there according to the criterion

$$w_g N_z N_y l(\omega - U_x k)\begin{cases} < 0 & \text{for transmission,} \\ > 0 & \text{for capture} \end{cases} \tag{3.5}$$

(cf. (1.1) and (2.10)).

As in §2 we find, on the wave-packet formalism, 'forbidden regions' for wave propagation. The z component m of the wavenumber is real only if

$$N_y^2 + N_z^2 \geqslant (\omega - U_x k)^2 \geqslant k^2 N_y^2/(k^2+l^2) \tag{3.6}$$

(cf. equation (2.11)). If a packet approaches a level $z = z_r$ where the wind takes any of these limiting values its z component of propagation speed decreases as $|z-z_r|^{\frac{1}{2}}$ and it is reflected.

Figure 2 provides four examples of the projection in the y, z plane of the path taken by a wave packet between generation (at one of the ringed points) and ultimate capture at one or other of the critical levels. The wind U_x is directed out of the paper and increases with z. All packets have the same positive ω and k. Those denoted by a plus sign have $l > 0$, while those denoted by a minus sign have a negative l of the same magnitude. The valve effect is again in evidence, and the sense in which it operates at each of the critical levels may readily be checked against (3.5). Note in particular the similarity between the (projected) path of the $l > 0$ packet around the lower critical level and that illustrated in figure 1 for the analogous hydromagnetic problem.

As in § 2 we now turn attention briefly to more general circumstances in which λ/L is not necessarily very small compared with unity, anticipating that *some* degree of transmission will then take place at *every* level. In contrast to the hydromagnetic problem discussed in § 2 the wave energy flux $\mathscr{E}$ per unit area in the z direction now depends on z owing to a continual exchange of energy between the wave and the mean flow. Nevertheless, it may readily be shown that between any two neighbouring levels of the seven singled out for special attention in figure 2 the quantity

$$Q \equiv \mathscr{E}/(\omega - U_x k) \tag{3.7}$$

(which is proportional to the rate of transfer of wind-directed momentum up the velocity gradient) *is* independent of z and accordingly acts as a convenient measure of the intensity of the wave (cf. Eliassen & Palm 1960; Booker & Bretherton 1967).

When $N_y = 0$ the valve effect disappears completely and all five levels

$$kU_x = \omega, \quad \omega \pm kN_y/(k^2+l^2)^{\frac{1}{2}}, \quad \omega \pm N_y \tag{3.8a, b, c}$$

merge into one. Booker & Bretherton (1967) have shown that provided that $N_z^2/(dU_x/dz)^2 \gtrsim 1$ the wave is then severely attenuated on transmission through such a level, viz.

$$\frac{Q_{\text{transmitted}}}{Q_{\text{incident}}} = \exp - 2\pi \left\{\frac{(k^2+l^2)\,N_z^2}{k^2(dU_x/dz)^2} - \frac{1}{4}\right\}^{\frac{1}{2}}. \tag{3.9}$$

(This is in fact a slight generalization of their original result, which was for the case $l = 0$.) Presumably when N_y is sufficiently small compared with ω the *net* attenuation suffered by a wave in its passage through *all five* levels (3.8*a, b, c*) is essentially that given by (3.9). This is certainly known to be the case in another simple modification of the basic ($N_y = 0$) system in which *rotation* with angular velocity Ω about a vertical axis gives rise to three singular levels

$$kU_x = \omega, \; \omega \pm 2\Omega, \tag{3.10}$$

as Jones (1967) has demonstrated numerically.

As far as the critical levels $kU_x = \omega \pm N_y$ *themselves* are concerned, an analysis similar to that in § 3 of A indicates that a wave described as being captured in the limit $\lambda/L \to 0$ is, in more general circumstances, partially transmitted, albeit with a transmission coefficient

$$\exp - 2\pi \left| \frac{l N_z}{k dU_x/dz} \right|_{z=z_c}. \tag{3.11}$$

This factor is independent of N_y and typically very small (but larger than the value given by (3.9) if $R > \frac{1}{4}$) provided only that the Richardson number $R \equiv N_z^2/(dU_x/dz)^2$ is greater than about unity. In accord with the wave-packet picture this transmission disappears completely when, other things being equal, $dU_x/dz \to 0$.

Note that if l^2/k^2 is sufficiently large the expressions (3.9) and (3.11) become identical and the levels (3.8*a*, *b*) merge. On the other hand, the one-way or valve attenuation (3.11) at each of the critical levels disappears when $l = 0$ (see also (3.5)), in which case the levels (3.8*b*, *c*) merge.

4. Concluding remarks

Valve-like critical-layer action, as encountered in a recent paper on the hydromagnetics of rotating fluids (Acheson 1972*a*), has been illustrated by two further examples, namely (i) hydromagnetic gravity wave propagation in a sheared magnetic field and (ii) internal gravity wave propagation in a shear flow. We have demonstrated this explicitly when $\lambda \ll L$ by means of an elementary approach using the concepts of wave packets and group velocity and have briefly summarized the results of more general treatments applicable even in certain circumstances for which λ/L is only marginally less than unity. When the wavelengths involved substantially exceed the length scales characteristic of inhomogeneities of the system, on the other hand, it is entirely possible that quite different phenomena occur. Substantial *reflexion* may then occur at critical levels and there may even be circumstances in which 'over-reflexion' (reflexion coefficient greater than unity) takes place and waves actually *extract* energy and momentum from the mean state (see, for example, Jones 1968; Breeding 1971; McKenzie 1972).

From a mathematical point of view these critical levels arise typically as singularities of second-order ordinary differential equations (see, for example, equation (2.6) of A). These singularities accordingly disappear when higher derivatives are introduced, as would occur in §2, for example, if either (*a*) dissipative effects were to be included or (*b*) the basic magnetic field were to have a rather more complicated form, e.g. $\mathbf{B}_0 = \{B_x(z), B_y(z), B_z\}$ (with $B_z \neq 0$). While, in view of the work of Hazel (1967) and Baldwin & Roberts (1970), one might anticipate that in smoothing out such singularities small *dissipative* effects will not significantly modify the overall properties of the critical layers predicted by the methods used here and in A, it is not yet clear that small *changes in the basic state* of the kind suggested above (even when $B_z \ll B_x(z_c)$, for example) will play such a passive and secondary role.

It is also important to bear in mind that we have yet to establish properly the circumstances in which the systems discussed in this paper and in A are *stable* to small disturbances. As far as §§2 and 3 are concerned this aspect of the problem has been studied so far only in the case of a *vertical* gradient of magnetic field/wind (for which the valve effect is absent). The hydromagnetic system of §2 is then stable regardless of the magnetic field profile $\mathbf{B}_0(z)$, while Miles (1961) and Howard (1961) have shown that the system in §3 is then stable provided that the Richardson number $R \equiv N_z^2/(dU_x/dz)^2$ everywhere exceeds 0·25. Even when similar sufficient conditions for stability (also on the basis of a *non-dissipative* theory) have been established for the more general case ($N_x^2 + N_y^2 \neq 0$) it will still be necessary to examine in §2, for example, the development of any 'resistive instabilities' (in the origin of which the critical layer plays an altogether different role; see, for example, Furth, Killeen & Rosenbluth 1963; Baldwin & Roberts 1972). Such an investigation will also be appropriate for the rotating system studied in A, for that is stable in the absence of dissipation regardless of both the details of the magnetic field profile $\mathbf{B}_0(z)$ and the angle between the z axis and $\mathbf{\Omega}$ (Acheson 1972*b*). It is important to bear in mind, however, that magnetic field configurations occurring naturally in rotating fluid systems of geophysical or astrophysical interest will be characterized (in contrast to the system discussed in A) by *curved* magnetic field lines, and that spatial variations of $\mathbf{B}$ can then lead to wavelike instabilities (see, for example, Acheson 1972*c*). Thus here again, as in the quite different context of the preceding paragraph, we are led to question not so much the validity of the neglect of dissipation but rather the typicality of the basic equilibrium configurations chosen for study.

The resolution of this question of whether or not the phenomena discussed in this paper and in A will be observable in practice only for a relatively narrow range of magnetic field/velocity configurations is, of course, essential before one can properly evaluate the possible role of these valve-like critical levels in natural systems. Moffatt (1972), for example, has recently indicated their practical interest from the point of view of turbulent fluid dynamos, where some selection mechanism producing a net flux of energy parallel to the rotation vector $\mathbf{\Omega}$ is vital to the whole regenerative process. Critical levels for internal gravity waves in a shear flow, on the other hand, are of interest in connexion with certain aspects of meteorology, although to avoid possible misunderstanding at this point it seems prudent to finally emphasize that developments in §3 are *not* explicitly directed toward this end (for critiques of the original (Bretherton 1966) model in relation to atmospheric conditions and various appropriate modifications to it see, for example, Bretherton 1969; Breeding 1971; Jones 1967; Lindzen 1970; Lindzen & Holton 1968) but rather toward the prediction of some *non-hydromagnetic* valve-like critical-level phenomena which may hopefully, at some stage, be tested experimentally (cf. Bretherton, Hazel, Thorpe & Wood 1967; see appendix to Hazel 1967). The extreme difficulty of performing hydromagnetic experiments to test *directly* the predictions of a 'perfectly conducting' theory such as that of A is well known.

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